§3.7 Rayleigh Flow
(One dimensional flow with heat addition)

Yi-Chao Xie
谢毅超
email yichao.xie@xjtu.edu.cn

西安交通大学　航天航空学院　工程力学系

2023 年 10 月 7 日

This note derives the governing equations for one dimensional flow with heat addition, also known as the Rayleigh flow. The model system considered is one dimensional flow (constant area flow) with heat added per unit mass being of q.

§3.7.1 Differential equations govered the flow property change for δq

Let's first derive equations describe the changes in flow properties when the flow is heated by a small amount of heat flux δq.

1. From the continuety equation, we have

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

Equation in differential form is $d(\rho u) = 0$

The density change:

$$\frac{d\rho}{\rho} = -\frac{du}{u} \tag{2}$$

2. From the momentum equation, we have

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{3}$$

In differential form, the equation can be written as

$$d(p + \rho u^2) = 0 \Rightarrow dp + ud(\rho u) + \rho u du = 0 \tag{4}$$

Plug in the continuity equation, cf $d(\rho u) = 0$, we have

$$dp + \rho u du = 0 \Rightarrow dp = -\rho u du \tag{5}$$

Plug in the EOS $\rho = \frac{p}{RT}$ and the definition of the Mach number $u^2 = M^2 a^2 = M^2 \gamma RT$ into equation (5), we have

The pressure change:

$$\frac{dp}{p} = -\gamma M^2 \frac{du}{u} \tag{6}$$

3. From the EOS $p = \rho RT$, we have

$$dp = R(Td\rho + \rho dT) \Rightarrow dT = \frac{dp}{R\rho} - \frac{Td\rho}{\rho} \Rightarrow \frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho} \tag{7}$$

Combing equation 6 and equation 2, we have

The temperature change:

$$\frac{dT}{T} = (1 - \gamma M^2)\frac{du}{u} \tag{8}$$

4. The entropy change when δq is added to the flow is

$$ds = c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho} \tag{9}$$

Substitute the expresses for $\frac{dp}{p}$ and $\frac{d\rho}{\rho}$ into the above equation, we have

The entropy change:

$$ds = c_v \gamma (1 - M^2) \frac{du}{u} \tag{10}$$

5. Now consider the total temperature change. From the definition of $T_0 = T + \frac{u^2}{2c_p}$, we have

$$dT_0 = dT + \frac{1}{c_p} u du \tag{11}$$

Since $c_p = \frac{\gamma R}{\gamma - 1}$, substitute this into equation , we have

$$dT_0 = dT + \frac{(\gamma - 1)T}{\gamma RT} u du = dT + (\gamma - 1) T M^2 \frac{du}{u} \tag{12}$$

Combing equations 9 and 13, we have

$$dT_0 = (1 - M^2) T \frac{du}{u} \tag{13}$$

Since $T_0 = T(1 + \frac{\gamma - 1}{2} M^2)$, we have

The total temperature change:

$$\frac{dT_0}{T_0} = \frac{1 - M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{du}{u} \tag{14}$$

6. For the change of the Mach number, consider $M = \frac{u}{\sqrt{\gamma RT}}$, we have

$$dM = M \frac{du}{u} - \frac{M}{2} \frac{dT}{T} \tag{15}$$

Substitute $\frac{dT}{T}$ into the above equation, we have

The Mach number change:

$$\frac{dM}{M} = \frac{1 + \gamma M^2}{2} \frac{du}{u} \tag{16}$$

We have derived changes of flow properties in terms of the relative velocity change $\frac{du}{u}$ when the flow is heated up by a small amount of δq. However, it is not straightforward to calculate $\frac{du}{u}$. We need to recast the above equations into a more easy to calculate way. Considering heating (cooling) the flow will result in a change in the total temperature, from the energy conservation equation, one has

$$T_{02} - T_{01} = \frac{q}{c_p} \tag{17}$$

For a very small increase in q, i.e., δq, we have $dT_0 = \frac{\delta q}{c_p}$. One can easily calculate the total temperature change. Let's now recast the above relations into functions of $\frac{dT_0}{T_0}$ using equation 15, i.e., the relation between $\frac{dT_0}{T_0}$ and $\frac{du}{u}$.

Flow properties change when heated by δq:

(1) The density change:

$$\frac{d\rho}{\rho} = -\frac{1 + \frac{\gamma-1}{2}M^2}{1 - M^2}\frac{dT_0}{T_0} \tag{18}$$

(2) The pressure change:

$$\frac{dp}{p} = -\gamma M^2 \frac{1 + \frac{\gamma-1}{2}M^2}{1 - M^2}\frac{dT_0}{T_0} \tag{19}$$

(3) The temperature change:

$$\frac{dT}{T} = (1 - \gamma M^2)\frac{1 + \frac{\gamma-1}{2}M^2}{1 - M^2}\frac{dT_0}{T_0} \tag{20}$$

(4) The entropy change:

$$ds = c_v \gamma (1 + \frac{\gamma - 1}{2}M^2)\frac{dT_0}{T_0} \tag{21}$$

(5) The velocity change:

$$\frac{du}{u} = \frac{1 + \frac{\gamma-1}{2}M^2}{1 - M^2}\frac{dT_0}{T_0} \tag{22}$$

(6) The Mach number change:

$$\frac{dM}{M} = \frac{1+\gamma M^2}{2}\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\frac{dT_0}{T_0} \tag{23}$$

Next, let's derive an equation for the total pressure change after the flow is heated with δq. Considering the entropy change across a normal shock, we have

$$ds = c_p\frac{dT_0}{T_0} - R\frac{dp_0}{p_0} \tag{24}$$

This equation can also be applied to flow with heat addition, except that $\frac{dT_0}{T_0}$ is zero for normal shock wave and it is nonzero for flow with heat addition. Let equation (22) equals to equation (25), one arrives at

$$c_v\gamma(1+\frac{\gamma-1}{2}M^2)\frac{dT_0}{T_0} = c_p\frac{dT_0}{T_0} - R\frac{dp_0}{p_0} \tag{25}$$

Rearrange the terms, recalling that $\gamma = \frac{c_p}{c_v}$, one obtains:

(7) the total pressure change:

$$\frac{dp_0}{p_0} = -\frac{\gamma}{\gamma-1}(\frac{\gamma-1}{2}M^2)\frac{dT_0}{T_0} = -\frac{\gamma}{2}M^2\frac{dT_0}{T_0} \tag{26}$$

Up to now, one can calculate flow properties change when a small amount of heat δq is added to the flow starting from location 1. Integrating equations (19-24) and (26), one can obtain the flow property changes when q is applied to the flow.

§3.7.2 **General flow property changes with heat addition**

Let's now discuss how the flow properties will change when heat is added to the flow. For ease of discussion, let's tabulate the changes in the following table.

Note the above relations will reverse for heat subtraction. Please verify this

	$T_0 \uparrow$, $q > 0$
$M < 1$	$\rho \downarrow$,$p \downarrow$,$s \uparrow$, $M \uparrow$, $T_0 \uparrow$, $p_0 \downarrow$, $u \uparrow$, $T \uparrow (M < \gamma^{-0.5})$,$T \uparrow (M > \gamma^{-0.5})$
$M > 1$	$\rho \uparrow$,$p \uparrow$,$s \uparrow$, $M \downarrow$, $T_0 \uparrow$, $p_0 \downarrow$, $u \downarrow$, $T \uparrow$

Table 1. Flow property change with heat addition.

yourself.

Now let's discuss the flow properties on the Rayleigh curve, i.e, in the $H - s$ curve or equivalently the $T - s$ curve. Combining equation 20 and 21, we have

$$\frac{dT}{ds} = \frac{1 - \gamma M^2}{\gamma(1 - M^2)} \frac{T}{c_v} \tag{27}$$

Several key points could be obtained:

(1) There are two branches, one for subsonic flow, one for the super sonic flow.

(2) Heating increases the entropy of the flow. Thus it drives the flow towards the sonic condition, i.e., increase the Mach number for a subsonic flow and decrease the Mach number for a supersonic flow.

(3) As $h \propto T$, heating the flow will also causes increase in h. But for subsonic flow, there is a region where the static temperature T decreases with heat addition. The reason is due to the internal energy being transferred into kinetic energy of the flow. This critical Mach number can be obtained from equation (21) by setting $\frac{dT}{T} = 0$. One obtains $M = \gamma^{-0.5}$.

(4) It is not possible to achieve supersonic flow by heat addition only. However, one can achieve supersonic flow by first heat up a subsonic flow to sonic condition and then cool it down to further increase its Mach number. The same method could be used to slow down a supersonic flow to subsonic flow, i.e, heat a supersonic flow to sonic condition and then cool it down to further decrease its Mach number.

§3.7.3 Integral forms of the flow properties change for heating up the flow with q.

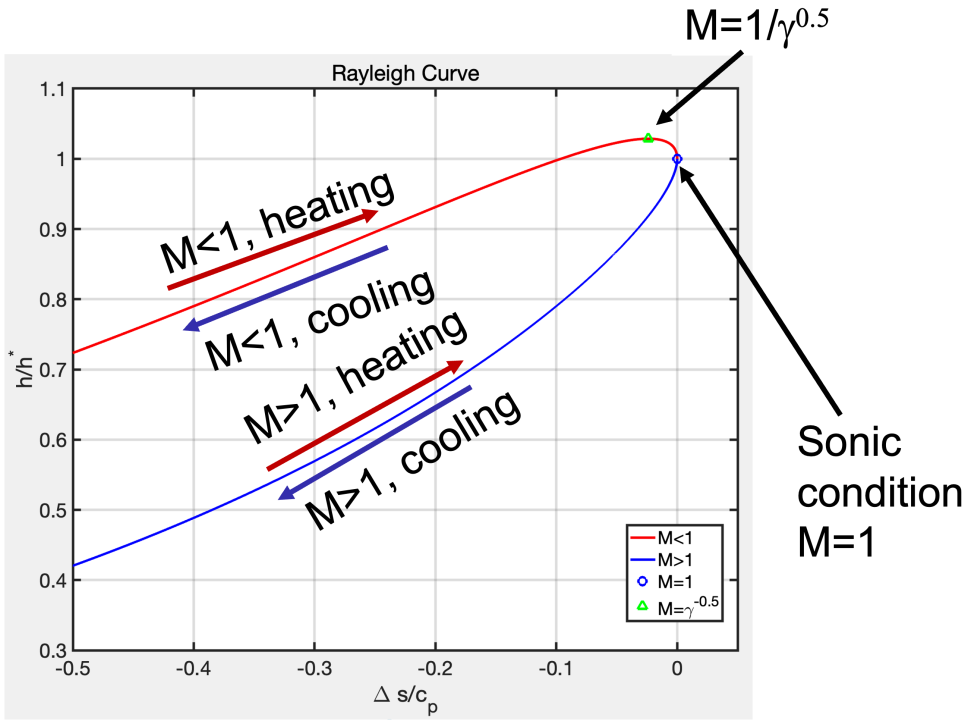

Figure 1. Rayleigh curve for one dimensional flow with heat addition.

Starting from the energy conservation equation, we have

$$q = c_p(T_{02} - T_{01}) \tag{28}$$

From the momentum equation, we have

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2 \tag{29}$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{30}$$

From the EOS, we have

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\frac{u_2}{u_1} \tag{31}$$

Considering the definition of the Mach number, one obtains

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2}{M_1}(\frac{T_2}{T_1})^{1/2} \tag{32}$$

Plug equation (32) into equation (31), we have

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{u_2}{u_1} = \frac{p_2}{p_1}\frac{M_2}{M_1}(\frac{T_2}{T_1})^{1/2} \tag{33}$$

thus,

$$\frac{T_2}{T_1} = (\frac{p_2}{p_1}\frac{M_2}{M_1})^2 = (\frac{1+\gamma M_1^2}{1+\gamma M_2^2})^2(\frac{M_2}{M_1})^2 \tag{34}$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2}(\frac{M_1}{M_2})^2 \tag{35}$$

For the total pressure ratio, we have

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = (1+\frac{\gamma-1}{2}M_2^2)^{\gamma/(\gamma-1)}(\frac{1+\gamma M_1^2}{1+\gamma M_2^2})(1+\frac{\gamma-1}{2}M_1^2)^{-\gamma/(\gamma-1)} \tag{36}$$

The above equation can be simplified as the following

$$\frac{p_{02}}{p_{01}} = (\frac{1+\gamma M_1^2}{1+\gamma M_2^2})(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2})^{\gamma/(\gamma-1)} \tag{37}$$

For the total temperature ratio, we have

$$\frac{T_{02}}{T_{01}} = \frac{T_{02}}{T_2}\frac{T_2}{T_1}\frac{T_1}{T_{01}} = (1+\frac{\gamma-1}{2}M_2^2)(\frac{1+\gamma M_1^2}{1+\gamma M_2^2})^2(\frac{M_2}{M_1})^2(1+\frac{\gamma-1}{2}M_1^2)^{-1} \tag{38}$$

The above equation can be simplified as

$$\frac{T_{02}}{T_{01}} = (\frac{1+\gamma M_1^2}{1+\gamma M_2^2})^2(\frac{M_2}{M_1})^2(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}) \tag{39}$$

As heating will increase the entropy of the flow. This increase in entropy can be calculated using

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{40}$$

$$s_1 - s_1 = c_p \ln[(\frac{1+\gamma M_1^2}{1+\gamma M_2^2})^2(\frac{M_2}{M_1})^2] - R\ln(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}) \tag{41}$$

Note that all the flow property ratios are functions of both M_1 and M_2 for a given gas (or γ). (Recall that the flow property ratio of a normal shock is only function of M_1 and γ.)

So how to solve a problem practically?
Given conditions in region 1, thus q and T_{01} can be obtained $\Rightarrow T_{02} \Rightarrow M_2 \Rightarrow p_2, T_2, \rho_2, p_{02}, T_{02}, \rho_{02}, ds$.
However, it is difficult to solve M_2 from equation (39) as $\frac{T_{02}}{T_{01}}$ is a complex function of M_2.

A more practical approach. Consider the example shown in figure 2. For the given condition in region 1, let the amount of heat needed to achieve $M_2 = M^* = 1$ state be q^*. If q_1 is added to the flow, it will change the flow properties at region 2, result in $M_2 = 1.5$, for example. Then starting from $M_2 = 1.5$, another heat q_1^* is added to achieve $M = 1$ state at the exit of the flow. One can see that the two states with M=1 are actually the same, i.e., for a given inlet flow condition, there is a reference state with $M^* = 1$ that is independent of how much heat is added to the system.

The flow properties between the inlet and the reference state can be easily obtained by setting $M_2 = 1$ in equations (30,34,35,37,39) and (41). The results are listed below:

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2} \tag{42}$$

$$\frac{T}{T^*} = M^2(\frac{1+\gamma}{1+\gamma M^2})^2 \tag{43}$$

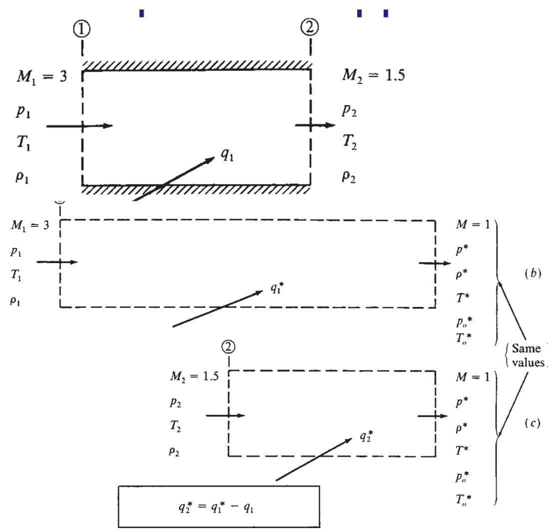

Figure 2. Illustration of the reference state with $M_2 = M^* = 1$.

$$\frac{\rho}{\rho^*} = \frac{1}{M^2}(\frac{1+\gamma M^2}{1+\gamma}) \tag{44}$$

$$\frac{T_0}{T_0^*} = \frac{(1+\gamma)M^2[2+(\gamma-1)M^2]}{(1+\gamma M^2)^2} \tag{45}$$

$$\frac{p_0}{p_0^*} = \frac{1+\gamma}{1+\gamma M^2}[\frac{2+(\gamma-1)M^2}{\gamma+1}]^{\frac{\gamma}{\gamma-1}} \tag{46}$$

Equations (42-46) are plotted in figure 3 for visualization.

Example questions. See textbook Examples 3.13 (p.106), 3.14 (p.107).

Questions: Could you tell the difference between the starred value defined here and that defined in §3.4 when we discussing the characteristic conditions?

§3.7.4 **Thermally chocking phenomena**

In this subsection, let discuss on the mass flow rate in the system. From equation (18), we have $\frac{q}{c_pT_{01}} = \frac{T_{02}}{T_{01}} - 1$. Substitute equation $\frac{T_{02}}{T_{01}} = (\frac{1+\gamma M_1^2}{1+\gamma M_2^2})^2(\frac{M_2}{M_1})^2(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2})$ into the above relation, one obtains

$$\frac{q}{c_pT_{01}} = (\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\frac{M_2}{M_1})^2(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}) - 1 \tag{47}$$

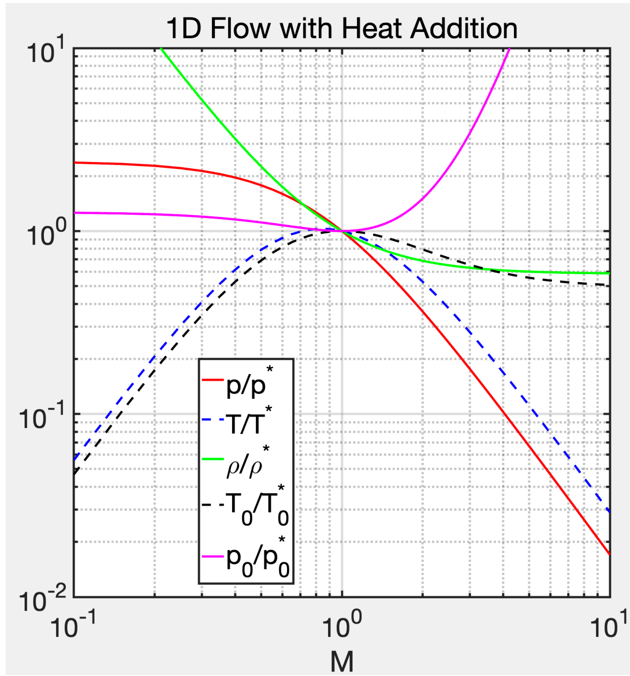

Figure 3. Change of flow properties for different Mach number.

Let $\frac{d(\frac{q}{c_p T_{01}})}{dM_2} = 0$, one obtains $M_2 = 1$. It can be proved that this corresponds to a maximum value of q allowed. In this case, we have

$$\frac{q_{max}}{c_p T_{01}} = [\frac{1+\gamma M_1^2}{(1+\gamma)M_1}]^2[\frac{1+\gamma}{2+(\gamma-1)M_1^2}] - 1 \tag{48}$$

Figure 4 plots $\frac{q_{max}}{c_p T_{01}}$ as a function of M_1. It is seen that q_{max} increases dramatically when M_1 decreases. That is to say small Mach number flow can absorb more heat.

What will happen when $q > q_{max}$?

For subsonic flow, the pressure wave propagates both downstream and upstream, causing a readjustment of the flow state. The main consequence is a reduced M_1, i.e., change the inlet condition, allowing a larger value of q. As a result, the mass flow rate will reduced as demonstrated in equation (49).

$$\dot{m}_1 = \rho_1 u_1 A = \frac{p_1}{RT_1}(M_1\sqrt{\gamma RT_1})A = \frac{p_1 M_1 \sqrt{\gamma}}{\sqrt{RT_1}}A \tag{49}$$

For supersonic flow, a shock wave will formed in the upstream position, resulting in a subsonic

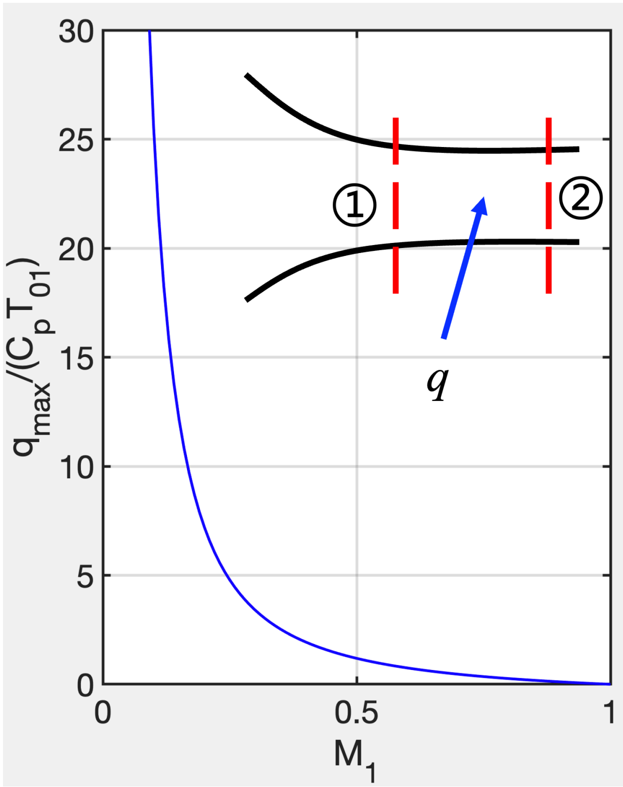

Figure 4. The maximum heat flux allowed without any change of the inlet condition vs M_1.

flow after the shock to allow more heat addition.

Thermal chocking is very important in the design of combustion engines as the thrust is directly related to the mass flow rate.

Example questions. See textbook Examples 3.15 (p.110), 3.16 (p.110).